Quantization of Spinor Field

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Quantization of real scalar field

1. Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2. \]

2. Euler-Lagrange equation (Klein-Gordon equation)

\[ (\partial^\mu \partial_\mu + m^2) \phi(x) = 0 \]

2. canonical momentum \( \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x) \), leading to the Hamiltonian density

\[ \mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = \frac{1}{2} \left[ (\partial_0 \phi)^2 + (\vec{\nabla} \phi)^2 \right] + \frac{1}{2} m^2 \phi^2. \]
quantization of real scalar field

3. Introducing commutation relation for $\phi$ and $\pi$

\[
[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}'),
\]
\[
[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0
\]

Heisenberg equation

\[
\dot{\phi}(\vec{x}, t) = i[H, \phi(\vec{x}, t)],
\]
\[
\dot{\pi}(\vec{x}, t) = i[H, \pi(\vec{x}, t)].
\]
4. Plane-wave expansion

\[ \phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} [a(\vec{k})e^{-ik \cdot x} + a^\dagger(\vec{k})e^{ik \cdot x}], \]

with \(\omega_k = \sqrt{\vec{k}^2 + m^2} \).

For \(\pi(x, t)\), we have

\[ \pi(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (-i\omega_k)[a(\vec{k})e^{-ik \cdot x} - a^\dagger(\vec{k})e^{ik \cdot x}], \]

\(a\) and \(a^\dagger\) can be expressed by the field operator

\[ a(\vec{k}) = i \int d^3x e^{ik \cdot x} \partial_0 \phi(x, t), \quad a^\dagger(\vec{k}) = -i \int d^3x e^{-ik \cdot x} \partial_0 \phi(x, t) \]
Commutation relation for $a$ and $a^\dagger$

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'),$$

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0.$$

Hamiltonian

$$H = \frac{1}{2} \int d^3 x \left[ \pi(x, t)^2 + |\nabla \phi(x, t)|^2 + m^2 \phi(x, t)^2 \right]$$

$$= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \omega_k [a(k)a^\dagger(k) + a^\dagger(k)a(k)].$$

Momentum

$$P = -\int d^3 x \pi(x, t) \nabla \phi(x, t) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \vec{k} [a(k)a^\dagger(k) + a^\dagger(k)a(k)].$$

Vacuum zero point energy

$$H = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \omega_k [a(k)a^\dagger(k) + a^\dagger(k)a(k)]$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \left[ \omega_k a^\dagger(k)a(k) + \frac{\omega_k}{2} \delta^3(0) \right]$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \omega_k [a^\dagger(k)a(k) + \frac{V}{2(2\pi)^3}].$$

The vacuum is not empty.

Infinity, but can be dropped.
Define Hamiltonian by normal ordering

\[
H = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \omega_k : [a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k})] :
\]

\[
= \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger(\vec{k}) a(\vec{k})
\]

Momentum

\[
\vec{P} = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \vec{k} a^\dagger(\vec{k}) a(\vec{k}).
\]

dfour-momentum

\[
P^\mu = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} k^\mu a^\dagger(\vec{k}) a(\vec{k}),
\]

\[
[P_\mu, a(\vec{k})] = -k_\mu a(\vec{k}), \quad [P_\mu, a^\dagger(\vec{k})] = k_\mu a^\dagger(\vec{k})
\]

Particle number operator

\[
N = \int \frac{d^3 k}{(2\pi)^2 2\omega_k} a^\dagger(\vec{k}) a(\vec{k}). \quad [N, P^\mu] = 0 \text{ for free field.}
\]
Can we do the same quantization as the scalar field case? I wish you read the chapter 3.5 of Peskin’s book, in which you will find a causality problem. In the following, I will adopt a simpler approach.
Spinor field Quantization

1. Lagrangian for free spinor field

\[ \mathcal{L} = \bar{\psi} \gamma^\mu i \partial_\mu \psi - m \bar{\psi} \psi, \]

This gives the equation of motion (free field)

\[ (i \partial - m) \psi = 0. \]

2. Conjugate momentum \( \pi_\psi \):

\[ \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \psi^\dagger, \]

Energy-Momentum Tensor and Momentum operator

\[ P_\nu = \int d^3 x T^{0\nu}(\pi, \psi), \quad T^{\mu\nu} = i \bar{\psi} \gamma^\mu \partial^\nu \psi. \]
3. Let us ASSUME the plane wave expansion:

\[ \psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_k} \sum_{\lambda=\pm 1} [b_\lambda(p)u_\lambda(p)e^{-ip\cdot x} + d_\lambda(p)v_\lambda(p)e^{ip\cdot x}], \]

\[ \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_k} \sum_{\lambda=\pm 1} [b_\lambda^\dagger(p)\bar{u}_\lambda(p)e^{ip\cdot x} + d_\lambda(p)\bar{v}_\lambda(p)e^{-ip\cdot x}], \]

The momentum operator can be written as:

\[ P_\mu = \int d^3x i\bar{\psi}(x)\gamma^0 \partial_\mu \psi(x) \]

\[ = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{\lambda=\pm 1} p_\mu [b_\lambda^\dagger(p)b_\lambda(p) - d_\lambda(p)d_\lambda^\dagger(p)] \]
Spinor field Quantization

4. Recall for a scalar field, we have

\[
\left[ P_\mu, a(k) \right] = -k_\mu a(k), \quad \left[ P_\mu, a^\dagger(k) \right] = k_\mu a^\dagger(k),
\]

we then may require

\[
\left[ P_\mu, b(k) \right] = -k_\mu b(k), \quad \left[ P_\mu, b^\dagger(k) \right] = k_\mu b^\dagger(k),
\]
\[
\left[ P_\mu, d(k) \right] = -k_\mu d(k), \quad \left[ P_\mu, d^\dagger(k) \right] = k_\mu d^\dagger(k).
\]

I have suppressed the polarization index.
The first relation is

$$\int \frac{d^3 p}{(2\pi)^3 2E_p} \sum_{\lambda=\pm 1} p_\mu [(b^\dagger_\lambda(p)b_\lambda(p) - d_\lambda(\vec{p})d^\dagger_\lambda(\vec{p}))], b_{\lambda'}(k)] = -k_\mu b_{\lambda'}(k).$$

which is equivalent to

$$\sum_{\lambda=\pm 1} [(b^\dagger_\lambda(p)b_\lambda(p) - d_\lambda(\vec{p})d^\dagger_\lambda(\vec{p}))], b_{\lambda'}(k)] = -(2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{p})b_{\lambda'}(\vec{k}).$$

Thus

$$\{b_\lambda(\vec{p}), b^\dagger_{\lambda'}(\vec{k})\} = (2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{p})\delta_{\lambda\lambda'}.$$  

$$\{d_\lambda(\vec{p}), d^\dagger_{\lambda'}(\vec{k})\} = (2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{p})\delta_{\lambda\lambda'}.$$
Using the relations,

\begin{align*}
\{ b_\lambda(p), b^\dagger_{\lambda'}(k) \} &= (2\pi)^3 2k^0 \delta^3(k - p) \delta_{\lambda\lambda'}, \\
\{ d_\lambda(p), d^\dagger_{\lambda'}(k) \} &= (2\pi)^3 2k^0 \delta^3(k - p) \delta_{\lambda\lambda'}.
\end{align*}

we have the quantization condition:

\begin{align*}
\{ \psi_\rho(x, t), i\psi^\dagger_\sigma(y, t) \} &= i\delta^3(x - y) \delta_{\rho\sigma}, \\
\{ \psi_\rho(x, t), \psi_\rho(y, t) \} &= \{ i\psi^\dagger_\sigma(x, t), i\psi^\dagger_\sigma(y, t) \} = 0,
\end{align*}
Spinor field Quantization

We summarize the spinor field quantization

- Writing down the Lagrangian density:
  \[ \mathcal{L} = \bar{\psi}(i\partial - m)\psi. \]

- Define the canonical momentum
  \[ \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger. \]

- Use the anti-commutation relation as the quantization condition:
  \[ \{ \psi_\rho(\vec{x}, t), i\psi_\sigma^\dagger(\vec{y}, t) \} = i\delta^3(\vec{x} - \vec{y})\delta_{\rho\sigma}, \]
  \[ \{ \psi_\rho(\vec{x}, t), \psi_\rho(\vec{y}, t) \} = \{ i\psi_\sigma^\dagger(\vec{x}, t), i\psi_\sigma^\dagger(\vec{y}, t) \} = 0, \]

- Plane wave expansion:
  \[
  \psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_k} \sum_{\lambda = \pm 1} [b_\lambda(\vec{p})u_\lambda(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + d_\lambda^\dagger(\vec{p})v_\lambda(\vec{p})e^{i\vec{p}\cdot\vec{x}}],
  \]
  \[
  \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_k} \sum_{\lambda = \pm 1} [b_\lambda^\dagger(\vec{p})\bar{u}_\lambda(\vec{p})e^{i\vec{p}\cdot\vec{x}} + d_\lambda(\vec{p})\bar{v}_\lambda(\vec{p})e^{-i\vec{p}\cdot\vec{x}}],
  \]
Momentum and angular momentum operator

\[ T_{\mu\nu} = i \bar{\psi} \gamma_\mu \partial_\nu \psi, \]
\[ P_\mu = \int d^3x T_{0\mu} = i \int d^3x \psi^\dagger \partial_\mu \psi, \]
\[ J_{\mu\nu\rho} = x_\nu T_{\mu\rho} - x_\rho T_{\mu\nu} + \bar{\psi} \gamma_\mu \frac{\sigma_{\nu\rho}}{2} \psi, \]
\[ M_{\mu\nu} = \int d^3x J_{0\mu\nu} = \int d^3x \psi^\dagger(x) \left[ i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right] \psi(x) \]

The last term is the spin operator.
Particle-Antiparticle

- We have the commutation relation:

\[
[P_\mu, \psi(x)] = i \int d^3y [\psi^\dagger(y) \partial_\mu \psi(y), \psi(x)]
\]

\[
= i \int d^3y (\psi^\dagger(y) \{\partial_\mu \psi(y), \psi(x)\} - \{\psi^\dagger(y), \psi(x)\} \partial_\mu \psi(y)
\]

\[
= -i \int d^3y \delta^3(\vec{x} - \vec{y}) \partial_\mu \psi(y) = -i \partial_\mu \psi(x),
\]

\[
[M^{\mu\nu}, \psi(x)] = \int d^3y [\psi^\dagger(y) \left( i(y_\mu \partial_\nu - y_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right) \psi(y), \psi(x)]
\]

\[
= - \left[ i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right] \psi(x).
\]
There are four different fermionic single particle states, and they are described by:

\[ b_\lambda^\dagger(\vec{p})|0\rangle, \quad d_\lambda^\dagger(\vec{p})|0\rangle \]

The subscript \( \lambda \) denotes the polarization, which is the eigenvalue of the helicity operator defined by momentum and Pauli-Lubanski operators.

How to distinguish the \( b_\lambda^\dagger(\vec{p}) \) and \( d_\lambda^\dagger(\vec{p}) \)?

Previously, I simply mentioned that they correspond to \textit{particle} and \textit{anti-particle}. Still we need a more concrete way.
Remember the U(1) symmetry in the Lagrangian,

\[ \mathcal{L} = \bar{\psi}(i\partial - m)\psi, \]

\[ \psi \rightarrow e^{-i\alpha}\psi, \]

This symmetry corresponds to a conserved current and a conserved charge:

\[ Q = \int \bar{\psi}(x)\gamma^0\psi(x) \]

\[ = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{\lambda=\pm 1} [b^\dagger_\lambda(\vec{p})b_\lambda(\vec{p}) - d^\dagger_\lambda(\vec{p})d_\lambda(\vec{p})] \]

notice this minus sign!
The charge operator commutate with the momentum operator:

\[
[Q, P_\mu] = 0.
\]

The commutation relation with the creation operators are

\[
[Q, b_\lambda^\dagger(\vec{p})] = b_\lambda^\dagger(\vec{p}), \quad [Q, d_\lambda^\dagger(\vec{p})] = -d_\lambda^\dagger(\vec{p}),
\]

\[
Q b_\lambda^\dagger(\vec{p}) |0\rangle = [Q, b_\lambda^\dagger(\vec{p})] |0\rangle = b_\lambda^\dagger(\vec{p}) |0\rangle,
\]

\[
Q d_\lambda^\dagger(\vec{p}) |0\rangle = [Q, d_\lambda^\dagger(\vec{p})] |0\rangle = -d_\lambda^\dagger(\vec{p}) |0\rangle,
\]

This implies that \( b_\lambda^\dagger(\vec{p}) |0\rangle \) and \( d_\lambda^\dagger(\vec{p}) |0\rangle \) are eigenstates of \( Q \) with the eigenvalue \( \pm 1 \).

The commutation relation with the field operators are

\[
[Q, \psi(x)] = -\psi(x), \quad [Q, \bar{\psi}(x)] = \bar{\psi}(x).
\]
propagator of real scalar field

\[ i\Delta(x - y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (e^{-ik \cdot (x-y)} - e^{ik \cdot (x-y)}) \]

\[ = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} (e^{-i\omega_k (x^0 - y^0)} - e^{i\omega_k (x^0 - y^0)}), \]

with \( \omega_k = \sqrt{|\vec{k}|^2 + m^2} \).

Represented by contour integral:

\[ \Delta(x) = \int_c \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik \cdot x}}{m^2 - k^2}. \]
Propagator of Spinor field

\[ C \]
Define retarded and advanced Green's functions

\[ \Delta_R(x) = -\frac{1}{2} (1 + \epsilon(x_0)) \Delta(x) = -\theta(x_0) \Delta(x), \]

\[ \Delta_A(x) = \frac{1}{2} (1 - \epsilon(x_0)) \Delta(x) = \theta(-x_0) \Delta(x), \]

\[ \Delta(x) = \Delta_A(x) - \Delta_R(x). \]
For retarded and advanced Green's functions:

\[\Delta_{R/A}(x) = \frac{dk^0}{2\pi} \int_{C_{R/A}} \frac{d^3 k}{(2\pi)^3} \frac{e^{-ik \cdot x}}{m^2 - k^2}.\]
Propagator of Spinor field

- **Time-ordered product:**

\[
\Delta_F(x - y) = -i \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle - i \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle \\
= -i \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} [\theta(x^0 - y^0) e^{-ik \cdot (x-y)} + \theta(y^0 - x^0) e^{ik \cdot (y-x)}] \\
= \int_{C_F} \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 - m^2} \\
= \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 - m^2 + i\epsilon}
\]
$C_F$
For a spinor, we have

\[
\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_\lambda u_a(p, \lambda) \bar{u}_b(p, \lambda) e^{-i p \cdot (x-y)}
\]

\[
= (i \partial_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i p \cdot (x-y)},
\]

\[
\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_\lambda v_a(p, \lambda) \bar{u}_b(p, \lambda) e^{-i p \cdot (x-y)}
\]

\[
= - (i \partial_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i p \cdot (x-y)}
\]
Dyson’s time-ordered product

\[
T\psi_a(x)\bar{\psi}_b(y) = \theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(y) - \theta(y^0 - x^0)\bar{\psi}_b(y)\psi_a(x),
\]

The vacuum expectation value is given as

\[
\langle 0| T\psi_a(x)\bar{\psi}_b(y)|0 \rangle = \left[ \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3 2E_p} (i\not{\partial}_x + m)_{ab} e^{-ip\cdot(x-y)} \right. \\
+ \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^3 2E_p} (i\not{\partial}_x + m)_{ab} e^{ip\cdot(x-y)} \right] \\
= (i\not{\partial}_x + m)_{ab} \\
\times \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \theta(x^0 - y^0) e^{-ip\cdot(x-y)} + \theta(y^0 - x^0) e^{ip\cdot(x-y)} \right] \\
= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{\partial} + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)},
\]
Dyson’s time-ordered product

\[ T\psi(x)\bar{\psi}(y) = \theta(x^0 - y^0)\psi(x)\bar{\psi}(y) + \theta(y^0 - x^0)(\bar{\psi}(y)^T \psi(x)^T)^T \]

Feynman’s propagation function

\[ iS_F(x - y) = \langle 0|T\psi(x)\bar{\psi}(y)|0 \rangle \]

\[ iS_F(x - y) = \int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{i(p + m)}{p^2 - m^2} e^{-ip \cdot (x-y)} \]

\[ = \int \frac{d^4p}{(2\pi)^4} \frac{i(p + m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \]
Derive the propagator for the spinor field

\[ \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i(p + m)}{p^2 - m^2 + i\epsilon}. \]