Quantization of Electromagnetic Field

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Review: Quantization of real scalar field

1. Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2. \]

2. canonical momentum \( \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x) \)

3. Commutation relation for \( \phi \) and \( \pi \)

\[ [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i \delta^3(\vec{x} - \vec{x}'), \]
\[ [\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \]

4. Plane-wave expansion \( \omega_k = \sqrt{\vec{k}^2 + m^2} \)

\[ \phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} [a(\vec{k})e^{-i\vec{k} \cdot \vec{x}} + a^\dagger(\vec{k})e^{i\vec{k} \cdot \vec{x}}] , \]
The commutation relation for $a$ and $a^\dagger$

$$
[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'),$
$$
[a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0.
$$

The time-ordered propagator is defined as:

$$
T[\phi(x)\phi(x')] = \theta(t - t')\phi(x)\phi(x') + \theta(t' - t)\phi(x')\phi(x),
$$
$$
(\Box_x + m)T[\phi(x)\phi(x')] = -i\delta^4(x - x').
$$

Then the vacuum expectation value is given as

$$
\langle 0|T[\phi(x)\phi(x')]|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{i}{p^2 - m^2 + i\epsilon}.
$$
1. Writing down the Lagrangian density:

\[ \mathcal{L} = \bar{\psi}(i\partial - m)\psi. \]

2. Define the canonical momentum

\[ \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger. \]

3. Use the anti-commutation relation as the quantization condition:

\[ \{ \psi_\rho(\vec{x}, t), i\psi_\sigma^\dagger(\vec{y}, t) \} = i\delta^3(\vec{x} - \vec{y})\delta_{\rho\sigma}, \]
\[ \{ \psi_\rho(\vec{x}, t), \psi_\rho(\vec{y}, t) \} = \{ i\psi_\sigma^\dagger(\vec{x}, t), i\psi_\sigma^\dagger(\vec{y}, t) \} = 0, \]

4. Plane wave expansion:

\[ \psi(x) = \int \frac{d^3 p}{(2\pi)^3 2E_k} \sum_{\lambda = \pm 1} [b_\lambda(\vec{p}) u_\lambda(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} + d_\lambda^\dagger(\vec{p}) v_\lambda(\vec{p}) e^{i\vec{p} \cdot \vec{x}}], \]
\[ \bar{\psi}(x) = \int \frac{d^3 p}{(2\pi)^3 2E_k} \sum_{\lambda = \pm 1} [b_\lambda^\dagger(\vec{p}) \bar{u}_\lambda(\vec{p}) e^{i\vec{p} \cdot \vec{x}} + d_\lambda(\vec{p}) \bar{v}_\lambda(\vec{p}) e^{-i\vec{p} \cdot \vec{x}}], \]
Nonzero commutation relation for creation and annihilation operators:

\[
\{ b_\lambda(\vec{p}), b_{\lambda'}^\dagger(\vec{k}) \} = \{ d_\lambda(\vec{p}), d_{\lambda'}^\dagger(\vec{k}) \} = (2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{p}) \delta_{\lambda\lambda'},
\]

and others are zero.

The time-ordered propagator \(^1\) is given as

\[
T\psi(x)\bar{\psi}(x') = \theta(t - t')\psi(x)\bar{\psi}(x') - \theta(t' - t)\bar{\psi}(x')\psi(x).
\]

The vacuum expectation value is defined as

\[
\langle 0| T\psi(x)\bar{\psi}(x') |0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{i(p + m)}{p^2 - m^2 + i\epsilon}.
\]

\(^1\)This is defined at the spinor component level
The Lagrangian density is given as

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

with the field strength tensor given as:

\[
F_{\mu\nu} = \begin{pmatrix}
0 & -E^1 & -E^2 & -E^3 \\
E^1 & 0 & -B^3 & B^2 \\
E^2 & B^3 & 0 & -B^1 \\
E^3 & -B^2 & B^1 & 0
\end{pmatrix}.
\]

The field strength tensor is directly related to the physical electric and magnetic field:

\[
E^i = -F^{0i}, \quad B^i = \frac{1}{2} \epsilon^{ijk} F^{jk}.
\]
The equation of motion is given as:

$$\partial_\mu F^{\mu\nu} = 0.$$  

which is equivalent to the Maxwell equation

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{B} = \frac{\partial}{\partial t} \vec{E}.$$  

We can define the dual of the field strength tensor:

$$\tilde{F}^{\mu\nu} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & -E^1 \\ B^3 & E^2 & E^1 & 0 \end{pmatrix}.$$  

Since due to the antisymmetric property, we have

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

This is equivalent to the other two components of Maxwell equation:

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}.$$
Electromagnetic field: Gauge Symmetry

- There is extra degrees of freedom in $A^\mu$

  \[ A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \alpha(x), \]

  with $\alpha(x)$ being any complex function. One can find that this does not change the $F^{\mu\nu}$, and thus corresponds to the same sets of physical fields.

- One can identify the above transformations as the extra degrees of freedom, or one may view them as the gauge symmetry.

- To remove the additional degrees of freedom, one usually uses the gauge fixing condition, for instance the Coulomb gauge:

  \[ \vec{\nabla} \cdot \vec{A} = 0, \quad A^0 = 0. \]

  Or the covariant Lorenz gauge (NOT Lorentz):

  \[ \partial_\mu A^\mu(x) = 0. \]
The canonical momentum is given as

$$\pi_\mu = \frac{\partial L}{\partial \dot{A}_\mu} = -F_{0\mu},$$

Apparently, $$\pi_0 = 0$$, and thus one can not quantize the $$A^0$$.

One can choose the gauge fixing condition to eliminate the unphysical degrees of freedom in $$A^\mu$$, such as the Coulomb gauge:

$$A^0 = 0, \quad \vec{\nabla} \cdot \vec{A} = 0.$$ 

This gauge is not covariant.
We introduce the Lagrangian density

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial^{\mu} \cdot A^{\mu})^2, \]

\(\alpha\) is an arbitrary constant (different with \(\alpha(x)\)). Setting \((\partial^{\mu} \cdot A^{\mu}) = 0\) (Lorenz gauge) will recover the original Lagrangian.

The canonical momentum is given as

\[ \pi_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}^{\mu}} = -F_{0\mu} - \frac{1}{\alpha} g_{0\mu} (\partial^{\nu} \cdot A^{\nu}). \]

Then \(\pi_0\) is not zero!

For simplicity, one can use \(\alpha = 1\), and this is called Feynman gauge (notice the two meanings of gauge!)
The quantization condition is chosen as:

\[
\begin{align*}
[A_\mu(x, t), \pi_\nu(y, t)] &= ig_{\mu\nu}\delta^3(\vec{x} - \vec{y}), \\
[A_\mu(x, t), A_\nu(y, t)] &= [\pi_\mu(x, t), \pi_\nu(y, t)] = 0.
\end{align*}
\]

This is equivalent to:

\[
\begin{align*}
[A_\mu(x, t), A_\nu(y, t)] &= [\dot{A}_\mu(x, t), \dot{A}_\nu(y, t)] = 0, \\
[A_\mu(x, t), \dot{A}_\nu(y, t)] &= -ig_{\mu\nu}\delta^3(\vec{x} - \vec{y}).
\end{align*}
\]
Electromagnetic Field Quantization: Gupta-Bleuler formalism

- Plane wave expansion

\[ A_\mu(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{\lambda=0}^3 \left[ a^{(\lambda)}(\vec{p}) \epsilon^{(\lambda)}_\mu(\vec{p}) e^{-ip \cdot x} + a^{(\lambda)^\dagger}(\vec{p}) \epsilon^{(\lambda)^*}_\mu(\vec{p}) e^{ip \cdot x} \right] , \]

Here the \( \epsilon_\mu \) are the polarization vectors. For massless particles, one may choose

\[ \epsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \quad \epsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \]

The left-handed and right-handed polarization is given as

\[ \epsilon^{(\pm)} = (\epsilon^{(1)} \pm \epsilon^{(2)})/\sqrt{2} . \]
The polarization vectors satisfy the orthonormality:

\[
\sum_{\lambda\lambda'} g_{\lambda\lambda'} \epsilon^{(\lambda)}_{\mu}(\vec{k}) \epsilon^{(\lambda')}_{\mu}(\vec{p}) = g_{\mu\nu},
\]

\[
\epsilon^{(\lambda)}_{\mu}(\vec{p}) \epsilon^{(\lambda')}_{\mu}(\vec{p}) = g^{\lambda\lambda'}.
\]

The creation and annihilation operators are expressed as:

\[
a^{(\lambda)}(\vec{p}) = i \int d^3x e^{ip\cdot x} \partial_0 \epsilon^{(\lambda)}_{\mu}(\vec{x}, t) A^\mu(\vec{x}, t),
\]

\[
a^{(\lambda)\dagger}(\vec{p}) = -i \int d^3x e^{-ip\cdot x} \partial_0 \epsilon^{(\lambda)}_{\mu}(\vec{x}, t) A^\mu(\vec{x}, t)
\]
The commutation relations for creation and annihilation operators

\[
[a^{(\lambda)}(\vec{p}), a^{(\lambda')}^{\dagger}(\vec{p}')] = -g^{\lambda\lambda'} 2p^0 (2\pi)^3 \delta^3(\vec{p} - \vec{p}'),
\]

\[
[a^{(\lambda)}(\vec{p}), a^{(\lambda')}^{\dagger}(\vec{p}')] = [a^{(\lambda')}^{\dagger}(\vec{p}), a^{(\lambda')}^{\dagger}(\vec{p}')] = 0.
\]

The above relations have a problem for the scalar polarization \(\lambda = 0\):

\[
\langle 0 | a^{(0)}(\vec{p}) a^{(0)}^{\dagger}(\vec{p}) | 0 \rangle < 0,
\]

on the other hand, this is a normalization of the state \(a^{(0)}^{\dagger}(\vec{p}) | 0 \rangle\) which must be larger than or equal to 0. This problem happens as we have included non-physical polarizations at the beginning.
To remove the non-physical polarizations, we put the constraints:

\[ \langle \psi | \partial^\mu A_\mu | \psi \rangle = 0. \]

This is a weakened Lorenz gauge fixing condition. Actually, the above constraint is equivalent to the following constraint:

\[ \partial^\mu A_\mu^{(+)} | \psi \rangle = 0, \]

where \( A_\mu^{(+)} \) is the positive energy component of \( A_\mu \).

For single particle states, apparently the transverse polarizations are physical:

\[
\begin{align*}
    i \partial^\mu A_\mu^{(+)} | \psi_T \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot x} \sum_{\lambda=0}^{3} a^{(\lambda)} (\vec{p}) \epsilon_{\mu}^{(\lambda)} (\vec{p}) p^\mu | \psi_T \rangle \\
    &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot x} \sum_{\lambda=1,2} a^{(\lambda)} (\vec{p}) \epsilon_{\mu}^{(\lambda)} (\vec{p}) p^\mu | \psi_T \rangle = 0.
\end{align*}
\]
Electromagnetic Field Quantization: Gupta-Bleuler formalism

- For scalar and longitudinal polarizations, we have
  \[
  \sum_{\lambda=0,3} p \cdot \epsilon^{(\lambda)}(\vec{p}) a^{(\lambda)}(\vec{p}) |\phi\rangle = 0
  \]
  namely the scalar polarization and longitudinal polarizations must exist simultaneously!

- The presence of scalar polarization and longitudinal polarizations will not affect the physical observables. Taking the expectation value of Hamiltonian as the example, we have
  \[
  \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \phi | \langle \psi_T | \int \frac{d^3p}{(2\pi)^3 2E_p} [\sum_{\lambda=1}^{3} a^{(\lambda)^\dagger}(\vec{p}) a^{(\lambda)}(\vec{p}) - a^{(0)^\dagger}(\vec{p}) a^{(0)}(\vec{p})] |\psi_T\rangle |\phi\rangle}{\langle \phi | \phi \langle \psi_T | \psi_T \rangle}
  \]
  \[
  = \frac{\langle \psi_T | \int \frac{d^3p}{(2\pi)^3 2E_p} [\sum_{\lambda=1}^{2} a^{(\lambda)^\dagger}(\vec{p}) a^{(\lambda)}(\vec{p})] |\psi_T\rangle}{\langle \psi_T | \psi_T \rangle}
  \]
The propagator of the electromagnetic field is given as

$$\langle 0| T A_\mu(x) A_\nu(y) |0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{-i}{p^2 + i\epsilon} g_{\mu\nu}$$

In a general gauge, we have

$$\langle 0| T A_\mu(x) A_\nu(y) |0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{-i}{p^2 i\epsilon} \left[ g_{\mu\nu} + (1 - \alpha) \frac{p_\mu p_\nu}{p^2 + i\epsilon} \right]$$

with the $\alpha$ being a constant.
Please derive the propagator for the electromagnetic field

\[ \langle 0| T A_\mu(x) A_\nu(y) |0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{-i}{p^2} \left[g_{\mu\nu} + (1 - \alpha) \frac{p_\mu p_\nu}{p^2}\right], \]

one may use the Feynman gauge with \( \alpha = 1 \).